Engineering fracture mechanics analysis of paper materials

Petri Mäkelä

KEYWORDS: Fracture, Failure, Notch, Crack, Strength, Web break, Tensile stiffness, Plasticity

SUMMARY: The aim of the present work was to develop an analytic fracture mechanics procedure that enables accurate predictions of failure of paper materials. Analytic expressions for prediction of the critical force and critical elongation of edge-notched paper webs were developed based on isotropic deformation theory of plasticity and \( J \)-integral theory. The analytic expressions were applied to predict the critical force and elongation of paper webs with different edge-notch sizes for six different paper materials. The accuracy of the analytic failure predictions was verified by numerical predictions and experiments on edge-notched paper webs, showing that the developed engineering fracture mechanics analysis procedure predicted failure accurately.


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PAPER materials commonly exhibit undesired fractures in manufacture, converting and end-use, but there is still no generally accepted fracture mechanics model available for predicting failures of notched paper structures. Previous work has shown that linear elastic fracture mechanics has limited applicability to paper materials (Sheath and Page 1974; Uesaka et al. 1979, Mäkelä and Östlund 1999, Mäkelä 2002), while non-linear fracture mechanics offers quantitative predictions of failure even when analysing small-sized defects in paper materials that exhibit pronounced non-linear material behaviour (Mäkelä and Östlund 1999; Wellmar et al. 2000, Mäkelä et al. 2009).

In a previous study (Mäkelä et al. 2009), a non-linear fracture mechanics model based on isotropic deformation theory of plasticity and \( J \)-integral theory is applied for predicting failure of paper materials. The fracture mechanics model is calibrated for six different paper materials based on laboratory test data, involving determination of the fracture toughness by finite element analysis of laboratory fracture toughness tests. The calibrated fracture mechanics models are used to numerically predict failure of large edge-notched paper webs for each of the investigated paper materials. The accuracy of the failure predictions are verified by tensile testing of edge-notched paper webs for the different investigated paper materials and several different notch sizes. The verification study shows that the numerical predictions of failure are in excellent agreement with the experiments. Consequently, the work by Mäkelä et al. (2009) shows that isotropic deformation theory of plasticity and \( J \)-integral theory constitute an accurate fracture mechanics modelling level for paper materials.

In a recent work (Mäkelä, Fellers 2012), a neat analytic procedure for calibrating the isotropic deformation theory of plasticity model based on laboratory tensile test data is developed. The analytic procedure is used to calibrate the isotropic theory of plasticity model for six different paper materials. The calibrated models are experimentally verified to accurately model the tensile behaviour of the investigated paper materials. In the same work, a closed-form analytic expression for determination of the fracture toughness of paper materials based on laboratory fracture toughness test data is developed. The analytic expression is used to determine the fracture toughness of the six investigated paper materials and is shown to determine the fracture toughness in excellent agreement with finite element analysis of laboratory fracture toughness tests. The work by Mäkelä and Fellers (2012) therefore shows that a non-linear fracture mechanics model can be calibrated without encountering numerical complexities.

The aim of the present work was to develop an analytic procedure for predicting failure of notched paper webs based on a calibrated non-linear fracture mechanics model. Such analytic procedure would enable engineering fracture mechanics analysis of paper materials, making it possible to predict failure of notched paper webs based on laboratory test data, without encountering numerical complexities.

Nomenclature

All equations in this work treat the in-plane mechanical behaviour of paper materials, assuming small deformation theory and plane stress conditions. All presented equations are based on the engineering stress format (force per unit width per unit thickness). The equations also apply to other stress formats, such as the line load format (force per unit width) or the specific stress format (force per unit width per unit grammage), provided that all used material parameters adhere to the same stress format convention. However, the equations involving the thickness \( \varepsilon \) to, in these equations should be set to unity when applying the line load format, while it should be substituted by the grammage, denoted by \( w \), when applying the specific stress format.

Materials and Methods

Materials

Six commercial grades of paper and board were investigated, viz. Fluting paper (Fluting), Sack paper (Sack), Newsprint (News), Testliner (Liner), Medium-weight coated paper (MWC), and Supercalendered paper (SC). The selected paper materials comprised widely different pulping conditions, papermaking strategies, and end-use requirements.
Table 1. Summary of the structural properties of the investigated paper materials.

<table>
<thead>
<tr>
<th>Property</th>
<th>Fluting</th>
<th>Sack</th>
<th>News</th>
<th>Liner</th>
<th>MWC</th>
<th>SC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grammage, ( w ) / g/m(^2)</td>
<td>111</td>
<td>79</td>
<td>45</td>
<td>100</td>
<td>90</td>
<td>51</td>
</tr>
<tr>
<td>Structural thickness, ( t ) / ( \mu )m</td>
<td>145</td>
<td>104</td>
<td>60.9</td>
<td>150</td>
<td>75.4</td>
<td>41.7</td>
</tr>
<tr>
<td>Structural density, ( \rho ) / kg/m(^3)</td>
<td>766</td>
<td>762</td>
<td>739</td>
<td>668</td>
<td>1194</td>
<td>1223</td>
</tr>
</tbody>
</table>

Table 2. Summary of the tensile material parameters and fracture toughness in MD of the investigated paper materials.

<table>
<thead>
<tr>
<th>Property</th>
<th>Fluting</th>
<th>Sack</th>
<th>News</th>
<th>Liner</th>
<th>MWC</th>
<th>SC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile stiffness index, ( E^w ) / MNm/kg</td>
<td>12.0</td>
<td>10.4</td>
<td>9.78</td>
<td>7.61</td>
<td>7.52</td>
<td>7.27</td>
</tr>
<tr>
<td>Strain-hardening modulus index, ( E_0^w ) / kNm/kg</td>
<td>518</td>
<td>376</td>
<td>217</td>
<td>181</td>
<td>216</td>
<td>176</td>
</tr>
<tr>
<td>Strain-hardening exponent, ( N )</td>
<td>3.29</td>
<td>3.43</td>
<td>4.65</td>
<td>4.35</td>
<td>3.69</td>
<td>4.18</td>
</tr>
<tr>
<td>Fracture toughness index, ( J_{cr}^w ) / Jm/kg</td>
<td>6.10</td>
<td>13.4</td>
<td>3.43</td>
<td>5.30</td>
<td>3.98</td>
<td>2.43</td>
</tr>
</tbody>
</table>

Table 3. Summary of the tensile material parameters and fracture toughness in CD of the investigated paper materials.

<table>
<thead>
<tr>
<th>Property</th>
<th>Fluting</th>
<th>Sack</th>
<th>News</th>
<th>Liner</th>
<th>MWC</th>
<th>SC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile stiffness index, ( E^w ) / MNm/kg</td>
<td>3.79</td>
<td>4.58</td>
<td>2.47</td>
<td>2.82</td>
<td>3.18</td>
<td>2.35</td>
</tr>
<tr>
<td>Strain-hardening modulus index, ( E_0^w ) / kNm/kg</td>
<td>59.7</td>
<td>194</td>
<td>34.8</td>
<td>53.5</td>
<td>45.3</td>
<td>30.0</td>
</tr>
<tr>
<td>Strain-hardening exponent, ( N )</td>
<td>7.06</td>
<td>2.58</td>
<td>7.60</td>
<td>4.99</td>
<td>5.40</td>
<td>6.32</td>
</tr>
<tr>
<td>Fracture toughness index, ( J_{cr}^w ) / Jm/kg</td>
<td>8.15</td>
<td>28.6</td>
<td>4.69</td>
<td>10.7</td>
<td>5.84</td>
<td>3.34</td>
</tr>
</tbody>
</table>

The paper materials were supplied as wrapped wound rolls from different European mills. The web width of the supplied rolls ranged between 0.95 m and 1.8 m. The rolls were unwrapped and the outermost paper layers were discarded. The unwrapped rolls were acclimatized two weeks in a controlled climate of 23°C and 50% RH before further actions were taken.

**Laboratory experiments**

Paper samples for laboratory experiments were collected from the supplied paper rolls. All paper samples were conditioned (ISO 187) before laboratory testing. The grammage (ISO 536) and the structural thickness and density (SCAN-P88:01) were determined. The structural properties of the investigated paper materials are summarised in Table 1.

Tensile testing and fracture toughness testing were performed in MD and CD for all investigated paper materials. The testing was performed following ISO 1924-3, with the exception that the fracture toughness testing was performed on centre-notched test pieces using an anti-buckling device. More detailed descriptions of the laboratory material testing are given in previous work (Mäkelä and Fellers 2012; Mäkelä et al. 2009).

**Material modelling**

The uniaxial tensile material behaviour of the investigated paper materials was modelled using an isotropic deformation theory of plasticity model that relates strain, \( \varepsilon \), to stress, \( \sigma \), as given by,

\[
\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{E_0}\right)^N \tag{1}
\]

where \( E \) denotes the tensile stiffness, \( E_0 \) is the strain-hardening modulus, and \( N \) denotes the strain-hardening exponent.

**Calibration of material model**

The material model in Eq 1 was calibrated based on the laboratory tensile test data. The tensile stiffness, \( E \), tensile strength, \( \sigma_r \), strain at break, \( \varepsilon_r \), and tensile energy absorption, \( W_r \), were determined following a procedure suggested by Mäkelä and Fellers (2012), which closely resembles ISO 1924-3. The strain-hardening exponent, \( N \), and the strain-hardening modulus, \( E_0 \), were determined using the following two analytic expressions, which were suggested by Mäkelä and Fellers (2012):

\[
N = \frac{\sigma_r^2 - 2EW_r}{\sigma_r^2 + 2W_r(E_0 - \sigma_r\varepsilon_r)} \tag{2}
\]

\[
E_0 = \frac{\sigma_r}{(E_0 - \sigma_r\varepsilon_r)^N}\tag{3}
\]

The determined tensile stiffness, strain-hardening exponent, and strain-hardening modulus in MD and CD are summarised in Tables 2-3 for the investigated paper materials.

**Determination of fracture toughness**

The fracture toughness, \( J_{cr} \), was determined based on the fracture toughness test data, using the following analytic expression that was suggested by Mäkelä and Fellers (2012),

\[
J_{cr} = 1.4096 \frac{a\sigma_{ns,cr}^2}{E} + \frac{a\sigma_{ns,cr}N+1}{E_0^a} \cdot \frac{N^{0.5494}}{0.2459N+0.4612} \tag{4}
\]

where \( a \) denotes the size of the notch in the fracture toughness test pieces after symmetry considerations (\( a=10 \) mm). The critical net-section stress, \( \sigma_{ns,cr} \), in Eq. 4 was evaluated from the fracture toughness test data using,

\[
\sigma_{ns,cr} = \frac{F_{cr}}{2t(W-a)} \tag{5}
\]
where \( F_{cr} \) denotes the critical force, i.e. the force at break in the fracture toughness test, \( t \) is the thickness of the paper material, and \( W \) is the width of the fracture toughness test piece after symmetry considerations (\( W = 25 \) mm). The determined fracture toughness in MD and CD are summarised in Tables 2-3 for the investigated paper materials.

**Tensile testing of large edge-notched paper webs**

The conditioned paper rolls were edge-trimmed in order to obtain a precise web width and to remove possible edge-damages caused by the previous transport and handling of the rolls. The web width after edge-trimming was 0.8 m for the fluting paper and sack paper, 0.95 m for the Medium-weight coated paper, and 1 m for the three remaining paper grades. The edge-trimming as well as the subsequent testing of the paper webs were performed in a controlled climate of 23°C and 50% RH.

The tensile testing of the large edge-notched paper webs was performed using the Wide Web Tensile Tester, a custom-built tensile tester developed at PFI in Norway (see Fig 1). The edge-trimmed paper web samples were mounted in the tensile tester using a clamping length of 1.88 m.

A sharp razor blade was used to introduce a notch in the edge of the paper web before the initiation of the test. The notch was oriented along CD and its location was alternated between the left and right edge of the paper web throughout the experimental trials, in order to reduce possible systematic influence of non-uniform material behaviour or skew loading on the test data. Several different notch sizes with lengths ranging from 1 mm to 40 mm were studied for each of the investigated paper materials. An anti-buckling guide, composed of two Plexiglas sheets that were placed on each side of the investigated paper materials, was used to prevent out-of-plane buckling of the notched region of the paper web during the tests.

All tests exhibiting obvious irregularities, such as skew mounting of the web or slippage in the clamping region, were rejected. Such irregularities, which caused non-uniform loading of the web, were detected by the formation of unsymmetrical web wrinkling patterns. The tensile testing of the notched paper webs are described in more detail in previous work (Mäkelä et al. 2009).

**Finite element analysis**

The commercial finite element code ABAQUS/Standard (ver 6.3) was used to analyse the tensile tests for the large edge-notched paper webs. An isotropic deformation theory of plasticity model, which extends Eq 1 to multiaxial conditions, was used to model the material behaviour. The used model establishes a one-to-one relation between the components of the strain tensor, \( \varepsilon_{ij} \), and the components of the stress tensor, \( \sigma_{ij} \), given by,

\[
\varepsilon_{ij} = S_{ijkl} \sigma_{kl} + \frac{3}{2} \left( \frac{\sigma_0}{K_0} \right)^N \frac{s_{ij}}{\sigma_0} \tag{6}
\]

where \( S_{ijkl} \) denote the components of the linear elastic compliance tensor, \( \sigma_0 \) is the von Mises effective stress, and \( s_{ij} \) denote the components of the deviatoric stress tensor. The parameters \( E_0 \) and \( N \) denote the strain-hardening modulus and strain-hardening exponent, respectively.

Three linear elastic material parameters, the tensile stiffness, \( E \), the Poisson’s ratio, \( \nu \), and the shear modulus, \( G \), are required to model the multiaxial linear elastic behaviour of isotropic materials. The tensile stiffness was collected from Tables 2-3, the Poisson's ratio was assumed to be 0.293 (Baum et al. 1981), and the shear modulus is defined by Eq 7. These three material parameters define the components of the linear elastic compliance tensor.

\[
G = \frac{E}{2(1+\nu)} \tag{7}
\]

The material parameters \( E_0 \) and \( N \), which were used to model the non-linear strain-hardening behaviour of the material, were also collected from Tables 2-3.

The numerical expense of the fracture mechanics analyses was reduced by utilising the symmetry in both geometry and loading of the large edge-notched paper webs by restricting the analysis to one half of the edge-notched paper web. Fig 2 shows an illustration of the type of finite element mesh that was used in the fracture mechanics analysis. The centre line in the figure indicates the symmetry cross section, where symmetry boundary conditions were applied. The loading imposed by the separation of the clamps in the tests of the large edge-notched paper webs was modelled by subjecting the

![Fig 1: The Wide Web Tensile Tester (PFI, Norway).](image1)

![Fig 2: Illustration of the used finite element mesh (edge-cracked rectangular panel with one half of the structure modelled after symmetry considerations).](image2)
nodes along the right vertical edge of the mesh to a uniform and monotonically increasing displacement in the y-direction and zero displacement in the x-direction. Isoparametric biquadratic eight-node plane stress elements with reduced integration were used in the geometrically linear finite element analysis.

The J-integral was evaluated using the implemented domain integral formulation in ABAQUS/Standard. Failure was predicted when the J-integral exceeded the fracture toughness of the material.

Fracture mechanics analysis was also utilised to evaluate the linear elastic and non-linear geometry functions of the introduced analytic expressions for predicting failure of the investigated large edge-notched paper webs.

Results

Semi-analytic expression for the J-integral of edge-notched panels

A semi-analytic expression for the J-integral of rectangular notched panels, which applies to mode I fracture of materials obeying the isotropic deformation theory of plasticity model in Eq 1, was presented by Mäkelä and Fellers (2012). The concerned expression is given by,

\[ J = \frac{a\sigma_{ns}^2}{2E} f_{el} \left( \frac{a}{W}; \frac{h}{W} \right) + \frac{a\sigma_{ns}^2}{2E} f_{nl} \left( \frac{a}{W}; \frac{h}{W}; N \right) \]  

where \( E \), \( E_0 \) and \( N \) are the material parameters of the material model in Eq 1, the parameters \( a \), \( W \) and \( h \) denote the in-plane characteristic dimensions (crack length, width, and length, respectively) of the notched panel, \( \sigma_{ns} \) is the net-section stress, \( f_{el} \) is a linear elastic geometry function that depends on the characteristic dimensions of the notched panel, and \( f_{nl} \) is a non-linear geometry function that depends on both the characteristic dimensions of the notched panel and the strain-hardening exponent, \( N \), of the material.

Fig 3 defines the in-plane characteristic dimensions of an edge-notched panel after symmetry considerations. The net-section stress for an edge-notched panel is defined by,

\[ \sigma_{ns} = \frac{F}{t(W-a)} \]  

where \( t \) is the thickness of the panel and \( F \) denotes the force applied to the panel.

Analytic procedure for predicting the critical force of edge-notched panels

This section treats the development of an analytic expression for predicting the critical force of edge-notched panels based on the semi-analytic expression for the J-integral in Eq 8. Initially, a 2 m long and 1 m wide rectangular panel (\( h=1 \) m, \( W=1 \) m) was considered. Four different edge-notch sizes, viz. 5 mm, 10 mm, 15 mm and 25 mm, were studied (\( a=5; 10; 15 \) and 25 mm).

The geometry functions of the expression in Eq 8 were evaluated by finite element analysis for each the four considered edge-notched panel geometries. The material model in Eq 6 was used, with both the tensile stiffness and the strain-hardening modulus set to one. The applied loading consisted of a monotonically increased mean strain that was ramped up to 100% in 100 equidistant steps.

The linear elastic geometry function, \( f_{el} \), was determined by using the linear elastic part of the material model, i.e. Eq 6 with the second term on the right-hand side set to zero. The linear elastic part of the J-integral expression, i.e. Eq 8 with the second term on the right-hand side set to zero, was then least squares fitted to the numerically obtained relation between the J-integral and the net-section stress, using the linear elastic geometry function as a free parameter. The evaluated linear elastic geometry functions for the four studied edge-notched panels are summarised in Table 4.

The non-linear geometry function was evaluated similarly, with the exceptions that the complete material model in Eq 6 was used to model the material behaviour and that the complete J-integral expression in Eq 8, with the evaluated value of the linear elastic geometry function from Table 4 inserted, was used in the least squares fitting. The non-linear geometry function was further evaluated for a number of different values of the strain-hardening exponent, in an interval enclosing the strain-hardening exponents of the investigated paper materials in the present work. The numerically evaluated relation between the non-linear geometry function and the strain-hardening exponent is presented as crosses in Fig 4 for each of the four studied edge-notched panel geometries.
A second-order polynomial expression, given by Eq 10, was least squares fitted to the numerically obtained relation between the non-linear geometry function and the strain hardening exponent for each of the studied edge-notched panels. The evaluated coefficients of the polynomial expression ($A_f$, $B_f$ and $C_f$) are summarised in Table 4.

$$f_{nl} = A_f N^2 + B_f N + C_f$$  \[10\]

The behaviour of the calibrated polynomial expression is presented in Fig 4, as a solid line for each of the studied edge-notched panel geometries, showing that the numerically obtained non-linear geometry functions were modelled excellently by Eq 10. An expression relating the $J$-integral to the net-section stress for a notched panel, such as Eq. 8, can also be used to express the relation between the critical value of the $J$-integral, i.e. the fracture toughness, and the critical net-section stress, i.e. the net-section stress at failure of the notched panel. By also utilising Eq 10, the semi-analytic expression for the $J$-integral in Eq 8 can be re-formulated as an analytic relation between the fracture toughness, $J_{cr}$, and the critical net-section stress, $\sigma_{ns,cr}$, as given by Eq 11. When the tensile material parameters ($E$, $E_0$, $N$) and the fracture toughness ($J_{cr}$) are known, Eq 10 enables prediction of the critical net-section stress for the studied edge-notched panels by utilising the geometry function parameters ($f_{EL}$, $A_f$, $B_f$, $C_f$) in Table 4.

$$J_{cr} = \frac{\sigma_{ns,cr}^2}{E_{EL}} - \frac{\sigma_{ns,cr}^2}{E_0} - \left( A_f \frac{a}{W} \right) N^2 + B_f \left( \frac{a}{W} \right) N + C_f \left( \frac{a}{W} \right)$$  \[11\]

The critical force, $F_{cr}$, i.e. the force at break of the edge-notched panel, may then be predicted based on the critical net-section stress, using the expression,

$$F_{cr} = \sigma_{ns,cr} t (W - a)$$  \[12\]

where $t$ denotes the thickness of the edge-notched panel.

Numerical verification of the analytic procedure for predicting critical force

The critical force was predicted analytically for edge-notched paper webs with geometries conforming to the four analysed panel geometries in the previous section. The predictions were performed by inserting the tensile parameters and fracture toughness from Tables 2-3 and the geometry parameters from Table 4 into Eq 11, for each combination of paper material, material direction, and panel geometry. The only remaining unknown parameter in Eq 11, i.e. the critical net-section stress, was then calculated using the Newton-Raphson method. Finally, the critical force was predicted based on the calculated critical net-section stress by using Eq 12.

The analytic predictions of the critical force were compared with predictions of the critical force obtained by finite element analysis. The results of this numerical verification study are presented in Fig 5, showing that the analytic predictions agreed excellently with the predictions obtained by finite element analysis. This result implies that the suggested analytic procedure can be used to predict the critical force of edge-notched paper webs with comparable accuracy as when using finite element analysis.

Semi-analytic expression for the compliance of notched panels

The compliance of a panel, i.e. the relation between the strain and stress, is altered when a notch is introduced in the panel. This section treats the development of a semi-analytic expression for the compliance of notched panels. The total strain, $\varepsilon$, of the uniaxial deformation theory of plasticity model in Eq 1 can be divided into an linear elastic part, $\varepsilon_{el}$, and a non-linear part, $\varepsilon_{nl}$, according to,

$$\varepsilon_{el} = \frac{\sigma}{E}$$  \[13\]

$$\varepsilon_{nl} = \left( \frac{\sigma}{E_0} \right)^N$$  \[14\]

The presence of a notch gives rise to non-uniform strain an stress fields in the panel, motivating that the apparent strain, $\varepsilon_{app}$, is introduced. The apparent strain is a measure of the mean strain of the panel, defined as the total elongation of the notched panel divided by its initial length. Dimensional analysis and $J$-integral theory can be utilised to derive the principal form of a semi-analytic expression for the apparent strain of notched panels. For a notched rectangular panel exhibiting mode I fracture, the principal form of the apparent strain for linear elastic materials obeying Eq 13 can be expressed as:
The corresponding principal form for non-linear materials obeying Eq 14 can be expressed as:

\[
\varepsilon_{\text{app,nl}} = \left( \frac{a}{W} \right)^N + \left( \frac{h}{W} \right)^N \varepsilon_{\text{nl}} \left( \frac{a}{W}, \frac{h}{W}; N \right)
\]  

[16]

In these two expressions, \(\sigma\) denotes the remotely applied stress, \(\Sigma\) is a stress measure characterising the severity of the loading, \(a, W\) and \(h\) denote the in-plane characteristic dimensions (crack length, width, and length, respectively) of the notched rectangular panel, \(g_{\text{el}}\) is a linear elastic geometry function that depends on the characteristic dimensions of the notched panel, while \(g_{\text{nl}}\) is a non-linear geometry function that depends on the characteristic dimensions of the notched panel and the strain-hardening exponent of the material.

An approximate semi-analytic expression for the apparent strain of notched panels, which applies to mode I fracture of materials obeying the isotropic deformation theory of plasticity model in Eq 1, can be formulated by adding the principal forms in Eq 15 and Eq 16, given

\[
\varepsilon_{\text{app}} = \frac{\sigma}{E} + \frac{\Sigma}{E} g_{\text{el}} \left( \frac{a}{W}, \frac{h}{W} \right) + \left( \frac{\Sigma}{E} \right)^N \varepsilon_{\text{nl}} \left( \frac{a}{W}, \frac{h}{W}; N \right)
\]  

[17]

The expression in Eq 17 relates the apparent strain to the behaviour of the material, the characteristic in-plane dimensions of the panel, and the applied loading.

**Analytic procedure for predicting critical elongation of edge-notched panels**

In a previous section, a procedure for predicting the critical force of edge-notched paper webs was developed. This section treats the development of an analytic expression for predicting the corresponding critical elongation of the edge-notched paper webs.

The definition of the in-plane characteristic dimensions of edge-notched panels in Fig 3 and the use of the net-section stress as the stress measure for characterising the severity of the loading were adopted, in analogy with the development of the analytic procedure for predicting the critical force. Furthermore, in order to avoid confusion by mixing two different stress measures in one expression, the remotely applied stress, \(\sigma\), in Eq 17 was reformulated in terms of the net-section stress, \(\sigma_{ns}\), by using the expression:

\[
\sigma = \sigma_{ns} \frac{W-a}{W}
\]  

[18]

When these measures were implemented, the expression in Eq 17 may be re-expressed as Eq 19.

The 2 m long and 1 m wide rectangular panels (\(h=1\) m, \(W=1\) m) with four different edge-notch sizes (\(a=5\); 10; 15 and 25 mm) were studied again. The geometry functions in Eq 19 were evaluated by revisiting the finite element analyses of the four panel geometries that were used to evaluate the geometry functions of the semi-analytic expression for the \(J\)-integral.

The linear elastic geometry function, \(g_{\text{el}}\), was evaluated by least squares fitting the linear part of Eq 19 (second term on the right-hand side set to zero) to the numerically obtained relation between the apparent strain and the net-section stress from the linear elastic analysis, using the linear elastic geometry function as a free parameter. The evaluated linear elastic geometry functions for the four studied edge-notched panels are summarised in Table 5.

<table>
<thead>
<tr>
<th>(a/W)</th>
<th>(g_{\text{el}})</th>
<th>(A_p)</th>
<th>(B_p)</th>
<th>(C_p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>-1.17E-02</td>
<td>4.57E-04</td>
<td>-1.10E-02</td>
<td>-2.09E-02</td>
</tr>
<tr>
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<td>-1.15E-02</td>
<td>4.92E-04</td>
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<td>-2.11E-02</td>
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<tr>
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<td>-1.12E-02</td>
<td>5.29E-04</td>
<td>-1.01E-02</td>
<td>-2.14E-02</td>
</tr>
<tr>
<td>0.025</td>
<td>-1.04E-02</td>
<td>5.99E-04</td>
<td>-8.61E-03</td>
<td>-2.23E-02</td>
</tr>
</tbody>
</table>

Table 5. Summary of the evaluated linear elastic geometry function, \(g_{\text{el}}\) and the evaluated coefficients \((A_p, B_p, C_p)\) of the second-order polynomial expression in Eq 20, for rectangular edge-notched panels characterised by \(h/W=1\).

The non-linear geometry function, \(g_{\text{nl}}\), was evaluated similarly, with the exceptions that the complete expression in Eq 19, with the linear elastic geometry function from Table 5 inserted, was fitted to the numerical data from the non-linear analysis. The non-linear geometry function was further evaluated for a number of different values of the strain-hardening exponent. The numerically evaluated relation between the non-linear geometry function and the strain-hardening exponent is presented as crosses in Fig 6 for each of the four studied edge-notched panel geometries.
\[ \varepsilon_{\text{app}} = \frac{\sigma_{\text{nl}}}{E} \left( 1 - \frac{a}{W} + g_{\text{el}} \left( \frac{a}{W}, \frac{h}{W} \right) \right) + \left( \frac{\sigma_{\text{nl}}}{E_0} \right)^N \left( \left( 1 - \frac{a}{W} \right)^N + g_{\text{nl}} \left( \frac{a}{W}, \frac{h}{W}, N \right) \right) \]  

\[ g_{\text{nl}} = A_g N^2 + B_g N + C_g \]  

\[ \varepsilon_{\text{app,cr}} = \frac{\sigma_{\text{nl,cr}}}{E} \left( 1 - \frac{a}{W} + g_{\text{el}} \left( \frac{a}{W}, \frac{h}{W} \right) \right) + \left( \frac{\sigma_{\text{nl,cr}}}{E_0} \right)^N \left( \left( 1 - \frac{a}{W} \right)^N + A_g \left( \frac{a}{W}, \frac{h}{W} \right) N^2 + B_g \left( \frac{a}{W}, \frac{h}{W} \right) N + C_g \left( \frac{a}{W}, \frac{h}{W} \right) \right) \]

A second-order polynomial expression, given by Eq 20, was least squares fitted to the numerically obtained relation between the non-linear geometry function and the strain hardening exponent for each of the studied edge-notched panels. The evaluated coefficients of the polynomial expression \((A_g, B_g \text{ and } C_g)\) are summarised in Table 5.

The behaviour of the calibrated polynomial expression is presented in Fig 5, as a solid line for each of the studied edge-notched panels, showing that the numerically obtained non-linear geometry functions were modelled excellently by Eq 20.

An expression relating the apparent strain to the net-section stress for a notched panel, such as Eq 19, can also be used to express the relation between the critical apparent strain and the critical net-section stress. By also utilising Eq 20, the semi-analytic expression for the apparent strain in Eq. 19 can be reformulated as an analytic relation between the critical apparent strain and the critical net-section stress, given by Eq 21. When the tensile material parameters \((E, E_0, N)\) and the critical net-section stress \((\sigma_{\text{nl,cr}})\) are known, Eq 21 enables the prediction of the critical apparent strain for the studied edge-notched panels by utilising the geometry function parameters \((g_{\text{el}}, A_g, B_g, C_g)\) in Table 5.

The critical elongation, \(\delta_{cr}\), i.e. the elongation at break of the edge-notched panel, may then be predicted based on the calculated critical apparent strain, using the expression,

\[ \delta_{cr} = 2\varepsilon_{\text{app,cr}} \]

**Numerical verification of the analytic procedure for predicting critical elongation**

In a previous section, the critical force of edge-notched paper webs was predicted analytically. This section treats the prediction of the corresponding critical elongation of the edge-notched paper webs.

The predictions were performed by inserting the tensile parameters from Tables 2-3, the geometry parameters from Table 5, and the previously predicted critical net-section stress into Eq 21, for each combination of paper material, material direction, and panel geometry. The critical apparent strain was calculated, followed by prediction of the critical elongation using Eq 22.

The analytic predictions of the critical elongation were compared with predictions of the critical elongation obtained by finite element analysis. The results of this numerical verification study are presented in Fig 7, showing that the analytic predictions agreed excellently with the predictions obtained by finite element analysis. This result implies that the suggested analytic procedure can be used to predict the critical elongation of edge-notched paper webs with comparable accuracy as when using finite element analysis.

**Experimental verification of the analytic procedures for predicting failure**

The accuracy of the developed analytic procedures for predicting failure of edge-notched paper webs were verified by experiments. The experiments comprised of large edge-notched paper webs in MD. The tested webs were 0.8 m, 0.95 m, or 1.0 m wide and a clamping length of 1.88 m. Different edge-notch sizes ranging between 1 and 40 mm were introduced in the paper webs prior to the testing.

Finite element analysis was used to determine the linear elastic and non-linear geometry functions for the considered paper web geometries in the experiments, following the previously described methods in this work. Fig 8 shows the evaluated relations between the non-linear geometry function of the J-integral expression and the strain-hardening exponent for different edge-notch sizes in a 1.88 m long and 0.95 m wide panel. The crosses in the figure indicate the numerically obtained values, while the solid lines show the behaviour of the calibrated expression in Eq 10. The corresponding results for the non-linear geometry function of the apparent strain expression are shown in Fig 9. The linear elastic and non-linear geometry functions were also evaluated for the 0.8 m and 1.0 m wide edge-notched panels.

The critical net-section stress of the edge-notched paper webs in the experiments was evaluated by using Eq 11, supported by the material parameters in Table 2, the characteristic dimensions of the considered paper webs and the corresponding geometry functions. The critical force was then predicted by inserting the critical net-section stress into Eq 12.

The critical apparent strain of the edge-notched paper webs in the experiments was evaluated by inserting the

![Fig 8. Relation between the non-linear geometry function of the J-integral expression in Eq 8 and the strain-hardening exponent for different edge-notch sizes (W=0.95 m; h=0.94 m). Crosses show data obtained by finite element analysis, while solid lines show the behaviour of Eq 10.](image-url)
Fig. 9. Relation between the non-linear geometry function of the apparent strain expression in Eq. 19 and the strain-hardening exponent for different edge-notch sizes \((W=0.95 \, m; h=0.94 \, m)\). Crosses show data obtained by finite element analysis, while solid lines show the behaviour of Eq. 20.

Critical net-section stress into Eq. 21, followed by prediction of the critical elongation using Eq. 22.

Fig. 10 shows a comparison of the predicted critical force versus the experimentally determined critical force for all the edge-notched paper webs in the experimental study. The corresponding results for the critical elongation are shown in Fig. 11.

The results in Figs. 10-11 show that the analytical predictions of critical force and critical elongation, respectively, agreed excellently with the experiments for all investigated paper materials and all investigated notch sizes.

**Discussion**

This work outlines an engineering fracture mechanics analysis procedure and shows that the fracture mechanics of paper materials can be accessed analytically without encountering numerical complexities.

However, the geometry functions of the presented analytic procedure need to be evaluated numerically, e.g. by means of finite element analysis. Consequently, the realisation of the engineering fracture mechanics analysis procedure relies on numerically pre-determined geometry functions.

One strategy for dealing with the numerical complexity to evaluate the geometry functions is to swallow the bitter pill and evaluate them once and for all for a great range of web geometries and strain-hardening exponents. The evaluated geometry functions can thereafter be tabulated and re-used in engineering fracture mechanics analysis of paper materials for all time, without ever having to encounter numerical complexities again. This strategy has been adopted by Innventia when developing the fracture mechanics computer program FractureLab, which utilises and interpolates among pre-determined geometry functions originating from several thousands of finite element analyses.

Another way to deal with the numerical complexity is to apply engineering fracture mechanics in a way that minimises the need of geometry functions. This can be realised e.g. by defining a notched reference panel geometry and evaluate the geometry functions for this panel only. As an example, the geometry functions for a 2 m long and 1 m wide panel containing a 10 mm edge-notch \((W=1 \, m; h=1 \, m; \text{ and } a=10 \, mm)\), are given in Eqs 23-26. These geometry functions make it possible to analytically predict the critical force and elongation of the notched reference panel geometry based on laboratory material testing.

\[
\begin{align*}
  f_{el} &= 3.7824 \\
  f_{nl} &= -4.6807 \cdot 10^{-5} N^4 + 3.4276 \cdot 10^{-3} N^3 - 1.0714 \cdot 10^{-1} N^2 + 2.0944N + 1.7856 \\
  g_{el} &= -1.1514 \cdot 10^{-2} \\
  g_{nl} &= 3.7978 \cdot 10^{-7} N^4 - 2.7479 \cdot 10^{-5} N^3 + 8.1531 \cdot 10^{-4} N^2 - 1.1945 \cdot 10^{-2}N - 1.9158 \cdot 10^{-2}
\end{align*}
\]

[23] [24] [25] [26]
The predicted critical force and elongation for the notched reference panel geometry can then be used to rank the fracture performance of different paper materials or to scan the effects of various process parameters and chemical additives on the fracture performance of the paper material, without encountering numerical complexities.

This article summarises the second and last part in the work of developing an engineering fracture mechanics analysis procedure for paper materials. The first part (Mäkelä, Fellers 2012) primarily address the development of an analytic expression for determination of the fracture toughness of paper materials. The combined results of these two articles are forming the basis for the ongoing development of an ISO Technical Specification on the determination of fracture toughness of paper and board.

Conclusions

Analytic expressions for predicting the critical force and critical elongation of edge-notched paper webs were developed. The expressions were used to predict the critical force and elongation of large edge-notched paper webs with different notch sizes for six different paper materials. The accuracy of the failure predictions was verified by comparison with numerical fracture mechanics analysis and experiments, showing that the developed engineering fracture mechanics analysis procedure predicted failure of edge-notched paper webs accurately.

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Literature